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ON THE GLOBAL SYMMETRY OF THE CLASSICAL KEPLER PROBLEM

THOMAS LIGON and MANFRED SCHAAF

Sektion Physik der Universität München, Theoretische Physik, München, F.R.D.

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A model for the classical Kepler problem is presented in which both the temporal evolution and the symmetry group act globally in a simple and canonical way. These actions are generated by the Hamiltonian function, the angular momentum and the Runge–Lenz vector. The symmetry group is SO(4) for negative and $SO(1,3)_0$ for positive energy.

Introduction

Until recently, the discussion of the inner symmetry of the classical Kepler problem has been restricted to the infinitesimal aspect. That is, one has discussed the additional constant of motion, the so-called *Runge–Lenz vector* whose components, together with those of the angular momentum, in the case of negative energy, form the Lie algebra of the orthogonal group SO(4) under the Poisson bracket. While in the quantum mechanical Kepler problem the global SO(4) symmetry has been explored in 1935/36 by Fock [6] and Bargmann [4] (for a more recent review see Bander, Itzykson [3]), the integral curves of the vector fields generated by the Runge–Lenz vector on the classical phase space were not investigated until recently. Bacry, Ruegg, and Souriau [2] in 1966 were the first, as far as we know, to give an explicit solution of the differential equations for the one-parameter transformation groups generated by the Runge–Lenz vector. In somewhat more detail these were discussed by Rogers [10] and, independently, by Ligon [8]. Roughly speaking, in these papers the SO(4)-action on the phase space is induced from some distorted action on a 3-sphere. With other applications in mind, Moser [9] has "regularized" the Kepler problem, that is, he has enlarged the phase space in such a way that the temporal evolution generates a global time flow, a situation which is otherwise precluded by the existence of collision orbits. The aim of our paper is to transform the Kepler problem in such a way that both the time flow and the SO(4) symmetry are globally re alized in a simple and canonical way. This is also done for positive energy and the group $SO(1,3)_0$. We have succeeded in unifying the viewpoints of regularization of the phase space and of an undistorted action of the symmetry group. We will comment on the case of vanishing energy briefly and not discuss it in detail.

After completion of our paper, we have received a copy of a talk by Souriau [11] in which basically the same results are derived, but by somewhat different methods.

1. Preparatory remarks

The configuration space of the Kepler problem is $Q := \boldsymbol{R}^3 \setminus \{0\}$, that is, a three-dimensional real Euclidean vector space with the origin, the so-called *collision point*, removed. We think of this space as a C^∞-manifold and define the C^∞ coordinate functions $\hat{q}_i$: $Q \to \boldsymbol{R}$ by means of $\hat{q}_i(x_1, x_2, x_3) := x_i$, $i = 1, 2, 3$. The phase space of the Kepler problem is then the total space T^*Q of the cotangent bundle π_Q^*: $T^*Q \to Q$ on the configuration space (cf. Abraham–Marsden [1], Godbillon [7]). Let $\mathscr{F}M$ denote the ring of real-valued C^∞-functions on a C^∞-manifold M. We will consider the following elements of $\mathscr{F}T^*Q$:

1. *The coordinate functions* q_i, p_i: $T^*Q \to \boldsymbol{R}$, $i = 1, 2, 3$, which are defined by $q_i(y) := \hat{q}_i \circ \pi_Q^*(y)$ and $y =: \sum_{i=1}^{3} p_i(y) d\hat{q}_i|\pi_Q^*(y)$ for $y \in T^*Q$. We will consider the q_i, $i = 1, 2, 3$, as components of an $\boldsymbol{R}^3$-valued function, which we will write in the form $\vec{q} = (q_1, q_2, q_3)$. We will do the same for other three component functions. We will often use the Euclidean scalar product in $\boldsymbol{R}^3$, which we will denote with a dot, e.g. $\vec{p} \cdot \vec{q} := \sum_{i=1}^{3} p_i q_i$. For the norm, we will simply write $q := |\vec{q}| = (\vec{q} \cdot \vec{q})^{1/2}$, etc.

2. *The Hamilton function of the Kepler problem*:

$$H := \frac{p^2}{2m} - \frac{\alpha}{q}, \qquad \alpha > 0,\ m > 0. \tag{1.1}$$

3. *The components of angular momentum*:

$$\vec{L} = \vec{q} \times \vec{p}, \qquad (\vec{q} \times \vec{p})_i := \sum_{j,k=1}^{3} \varepsilon_{ijk} q_j p_k, \qquad i = 1, 2, 3, \tag{1.2}$$

where ε_{ijk} is the Levi–Civita symbol, that is, ε_{ijk} is totally antisymmetric and $\varepsilon_{123} = 1$.

4. *The components of the Runge–Lenz vector*:

$$\vec{R} = \vec{p} \times \vec{L} - \frac{\alpha m}{q} \vec{q}. \tag{1.3}$$

The canonical symplectic form on the phase space is

$$\omega = \sum_{i=1}^{3} dq_i \wedge dp_i. \tag{1.4}$$

The Hamiltonian vector field X_f belonging to a function $f \in \mathscr{F}T^*Q$ is defined by

$$df =: \omega(X_f, \cdot\,). \tag{1.5}$$

Then we have

$$X_f = \sum_{i=1}^{3} \left(\frac{\partial f}{\partial p_i} E_{q_i} - \frac{\partial f}{\partial q_i} E_{p_i} \right), \tag{1.6}$$

where $E_.$ is the unit vector field in the coordinate direction given by the index. Then the Poisson bracket $\{\cdot, \cdot\}$: $\mathscr{F}T^*Q \times \mathscr{F}T^*Q \to \mathscr{F}T^*Q$ has the form

$$\{f, g\} := \omega(X_f, X_g) = \sum_{i=1}^{3} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i} \right), \tag{1.7}$$

$f, g \in \mathscr{F}T^*Q$. $\mathscr{F}T^*Q$, considered as a real vector space, has, with the Poisson bracket, the structure of a Lie algebra. For the above-mentioned functions, one can calculate the following Poisson brackets

$$\begin{aligned}
&(\mathrm{I}) \quad \{q_i, H\} = \frac{\partial H}{\partial p_i} = \frac{p_i}{m}, \quad \{p_i, H\} = -\frac{\partial H}{\partial q_i} = -\frac{\alpha}{q^3} q_i, \quad i = 1, 2, 3, \\
&(\mathrm{II}) \quad \{L_i, H\} = 0, \quad \{R_i, H\} = 0, \quad i = 1, 2, 3, \\
&(\mathrm{III}) \quad \{L_i, L_j\} = \sum_{k=1}^{3} \varepsilon_{ijk} L_k, \quad \{L_i, R_j\} = \sum_{k=1}^{3} \varepsilon_{ijk} R_k, \\
&\qquad \{R_i, R_j\} = -2mH \sum_{k=1}^{3} \varepsilon_{ijk} L_k, \quad i, j = 1, 2, 3.
\end{aligned} \tag{1.8}$$

We will now comment on each of these groups of equations.

1. The Hamilton function H generates the temporal evolution of the system. The first group of equations in (1.8) is thus identical with the usual Hamiltonian equations of motion of the Kepler problem. The solutions of these equations are the coordinate functions of the integral curves of the vector field X_H belonging to H. Since we have removed the collision point O from the configuration space, and since the collision orbits with angular momentum $\vec{L} = \vec{O}$ go into this point for finite times, the vector field X_H is not complete. Thus, time does not act on the phase space as a one parameter group of global diffeomorphisms. This handicap can be overcome by embedding the phase space in a larger manifold, on which time does act as a global group of transformations. This so-called *regularization* of the Kepler problem was carried out by Moser [9] for the surfaces of constant negative energy in phase space. It is practically a matter of compensating the divergence of the potential $-\alpha/q$ at $q = O$ by $\vec{p}$ values in a sphere with infinite radius. The mapping of the phase space which we will describe also yields a regularization of the Kepler problem in which the surfaces of constant negative energy are of the same topological type as those of Moser [9].

2. Group (II) of equations in (1.8) shows that the functions L_i and R_i, $i = 1, 2, 3$, are constant along the orbits of H. The orbits of H thus lie in the inverse image of a value

of these functions. Because of the condition $\sum_{i=1}^{3} L_i R_i = 0$, which holds on the whole phase space, at most five of these constants of motion are independent. The surfaces of a constant value of the functions L_i and R_i, $i = 1, 2, 3$, are thus at least one-dimensional. One can show that they are exactly one-dimensional and, in fact, the orbits of H can immediately be given, explicitly, with the help of the constants of motion. The orbit belonging to the values l_i, r_i, of L_i, R_i, $i = 1, 2, 3$, is: $\{y \in T^*Q \mid \vec{q}(y) \cdot \vec{l} = O,\ \vec{q}(y) \cdot \vec{r} = l^2 - \alpha m q(y),\ \vec{p}(y) \cdot \vec{l} = O,\ \vec{p}(y) \cdot \vec{r} = -\alpha m/(l^2 q(y)) \vec{q}(y) \cdot (\vec{l} \times \vec{r})\}$. The first two conditions are the equations for the Keplerian planetary orbits in configuration space.

3. For the discussion of group (III) of equations in (1.8) we introduce a notation for the constant energy hypersurfaces in phase space:

$$\Sigma_E := \{y \in T^*Q \mid H(y) = E\}. \tag{1.9}$$

Since the 1-form dH has no zeroes on T^*Q, a general theorem (cf. Dieudonné [5], § 16.8.9, p. 42) tells us that the energy surfaces are closed submanifolds of T^*Q. We also introduce the following open submanifolds of T^*Q:

$$\Sigma_\pm := \bigcup_{E \gtrless 0} \Sigma_E. \tag{1.10}$$

Thus $T^*Q = \Sigma_- \cup \Sigma_0 \cup \Sigma_+$, and one can consider Σ_0 as the common boundary of Σ_+ and Σ_-. On Σ_ε, instead of L_i and R_i, we introduce:

$$L_i^\varepsilon := L_i|\Sigma_\varepsilon, \qquad K_i^\varepsilon := (2mH\varepsilon)^{-1/2} R_i|\Sigma_\varepsilon, \qquad \varepsilon = \pm,\ i = 1, 2, 3. \tag{1.11}$$

Obviously, $L_i^\varepsilon, K_i^\varepsilon \in \mathscr{F}\Sigma_\varepsilon$. Since H commutes with L_i and R_i, we have the following Poisson brackets on Σ_ε:

$$\begin{aligned} \{L_i^\varepsilon, L_j^\varepsilon\} &= \sum_{k=1}^{3} \varepsilon_{ijk} L_k^\varepsilon, \qquad \{L_i^\varepsilon, K_j^\varepsilon\} = \sum_{k=1}^{3} \varepsilon_{ijk} K_k^\varepsilon, \\ \{K_i^\varepsilon, K_j^\varepsilon\} &= -\varepsilon \sum_{k=1}^{3} \varepsilon_{ijk} L_k^\varepsilon. \end{aligned} \tag{1.12}$$

Under the Poisson bracket, the functions L_i^ε and K_i^ε generate the Lie algebras of the groups SO(4) and SO(1, 3)$_0$ on Σ_ε for $\varepsilon = -$ and $\varepsilon = +$, respectively.

4. Since $H = 0$ on Σ_0, the functions L_i and R_i in group (III) of equations (1.8) seem to generate the Lie algebra of the Euclidean space group ISO(3). However, that is not the case since, for the Hamiltonian vector fields belonging to the functions R_i, the following holds:

$$[X_{R_i}, X_{R_j}] = -X_{\{R_i, R_j\}} = -X_{-2mH \sum_{k=1}^{3} \varepsilon_{ijk} L_k} \to 2m \sum_{k=1}^{3} \varepsilon_{ijk} L_k X_H$$

for $H \to 0$. But $X_H \neq 0$ on Σ_0.

2. The negative energy case

On Σ_-, we consider the following eight real-valued functions:

$$\begin{aligned}
\xi_0 &= \frac{(-2mH)^{1/2}}{m\alpha}\vec{q}\cdot\vec{p}\sin\varphi+\left(\frac{qp^2}{m\alpha}-1\right)\cos\varphi,\\
\vec{\xi} &= \left(\frac{\vec{q}}{q}-\frac{\vec{q}\cdot\vec{p}}{m\alpha}\vec{p}\right)\sin\varphi+\frac{(-2mH)^{1/2}}{m\alpha}q\vec{p}\cos\varphi,\\
\eta_0 &= -\vec{q}\cdot\vec{p}\cos\varphi+\frac{m\alpha}{(-2mH)^{1/2}}\left(\frac{qp^2}{m\alpha}-1\right)\sin\varphi, \qquad (2.1)\\
\vec{\eta} &= -\frac{m\alpha}{(-2mH)^{1/2}}\left(\frac{\vec{q}}{q}-\frac{\vec{q}\cdot\vec{p}}{m\alpha}\vec{p}\right)\cos\varphi+q\vec{p}\sin\varphi,\\
\varphi &:= (-2mH)^{1/2}/(m\alpha)\vec{q}\cdot\vec{p}.
\end{aligned}$$

In this section, we pretend that the functions q_i, p_i and H are only defined on Σ_-, in order to avoid the technically correct but cumbersome notation of restricted functions $q_i|\Sigma_-$, $p_i|\Sigma_-$ and $H|\Sigma_-$. The functions (2.1) are obviously of the class C^∞. We now consider the six-dimensional C^∞-submanifold of $\boldsymbol{R}^4\times\boldsymbol{R}^4$ which is defined by

$$M_- := \{(\xi,\eta)\in\boldsymbol{R}^4\times\boldsymbol{R}^4|\ \xi^2 = 1,\ \xi\cdot\eta = 0,\ \eta^2 > 0\}, \qquad (2.2)$$

whereby we denote the Euclidean scalar product in $\boldsymbol{R}^4$ by

$$\xi\cdot\eta := \sum_{\alpha=0}^{3}\xi_\alpha\eta_\alpha = \xi_0\eta_0+\vec{\xi}\cdot\vec{\eta}$$

and the square of the Euclidean norm in $\boldsymbol{R}^4$ by $\xi^2 = \xi\cdot\xi$. M_- is obviously diffeomorphic to the total space $(T^*S^3)^\times$ of the cotangent bundle on the 3-sphere S^3, from which the zero section has been removed. Since T^*S^3 is trivial, M_- is also diffeomorphic to $S^3\times(\boldsymbol{R}^3\setminus\{0\})$. If one considers the functions defined in (2.1) as one $\boldsymbol{R}^4\times\boldsymbol{R}^4$-valued mapping of Σ_-, it is easily verified that the image of this mapping lies in M_-. The absence of the zero section corresponds to the condition

$$\eta^2 = -m\alpha^2/(2H) > 0. \qquad (2.3)$$

We define two C^∞-submanifolds of M_- by

$$N_\infty := \{(\xi,\eta)\in M_-|\xi_0 = 1\}, \qquad N_- := M_-\setminus N_\infty. \qquad (2.4)$$

N_∞ can be identified with the punctured cotangent space at the "north pole" of the 3-sphere and is thus diffeomorphic to $\boldsymbol{R}^3\setminus\{0\}$. N_- is open and dense in M_-.

THEOREM 1. *The functions defined in* (2.1) *map* Σ_- *diffeomorphically onto* N_-.

Proof: We have already mentioned that (2.1) yields a C^∞-mapping F from Σ_- into M_-. We will show that the image of F does not intersect N_∞. Because of $\xi\cdot\eta = 0$, η_0

would have to be zero on N_∞. According to (2.1), the equations $\xi_0 = 1$ and $\eta_0 = 0$ are identical with $\varphi \sin\varphi + (1+2qH/\alpha)\cos\varphi = 1$ and $\varphi\cos\varphi - (1+2qH/\alpha)\sin\varphi = 0$. If we multiply the first equation by $\sin\varphi$, the second by $\cos\varphi$ and add, we get $\varphi = \sin\varphi$ and thus $\varphi = 0$. The first equation then yields $qH = 0$, which cannot be true on Σ_-.

In order to show that $F\colon \Sigma_- \to N_-$ is a diffeomorphism, we construct a C^∞-mapping $G\colon N_- \to \Sigma_-$ for which $G \circ F = \mathrm{id}_{\Sigma_-}$ and $F \circ G = \mathrm{id}_{N_-}$ hold. To this end, we first show that the equation

$$\psi = \xi_0 \sin\psi - (\eta^2)^{-1/2}\eta_0 \cos\psi \tag{2.5}$$

defines a C^∞-function $\psi\colon N_- \to \boldsymbol{R}$, for which $|\psi(\xi, \eta)| \leqslant 1$ holds on all of N_-. We only have to prove that, for every $(\xi, \eta) \in N_-$, there exists one and only one ψ such that $-1 \leqslant \psi \leqslant 1$ and ψ solves (2.5). Then it follows immediately from the implicit function theorem that this solution ψ is a C^∞ function of (ξ, η). We write (2.5) in the form

$$\psi = (\xi_0^2 + \eta_0^2/\eta^2)^{1/2}\cos(\psi - \gamma), \tag{2.6}$$

with a suitable phase γ. The graphs of $f(\psi) = \psi$ and $g(\psi) = (\xi_0^2+\eta_0^2/\eta^2)^{1/2}\cos(\psi-\gamma)$ have exactly one intersection since $(\xi_0^2+\eta_0^2/\eta^2)^{1/2} \leqslant 1$, which follows from (2.2) with $\xi_0^2+\eta_0^2/\eta^2 \leqslant \xi_0^2+\eta_0^2/\eta^2+(\vec{\xi}\times\vec{\eta})^2/\eta^2 = 1$. Using (2.6), $|\psi(\xi, \eta)| \leqslant 1$ clearly holds. We now map N_- into $\boldsymbol{R}^3 \times \boldsymbol{R}^3$ by means of

$$\begin{aligned}
\vec{q} &= \frac{\eta^2}{m\alpha}\,[(\xi_0 - \cos\psi)\,(\eta^2)^{-1/2}\vec{\eta} - \big((\eta^2)^{-1/2}\eta_0 - \sin\psi\big)\vec{\xi}],\\
\vec{p} &= \frac{m\alpha[(\eta^2)^{-1/2}\vec{\eta}\sin\psi + \vec{\xi}\cos\psi]}{(\eta^2)^{1/2}[1-\xi_0\cos\psi - (\eta^2)^{-1/2}\eta_0\sin\psi]},\\
\psi &= \xi_0\sin\psi - (\eta^2)^{-1/2}\eta_0\cos\psi,
\end{aligned} \tag{2.7}$$

whereby ψ is the C^∞ function on N_- defined by the last equation. The denominator of the equation for $\vec{p}$ cannot become zero. Otherwise, one would have

$$\xi_0\cos\psi + (\eta^2)^{-1/2}\eta_0\sin\psi = 1 \quad \text{and} \quad \xi_0\sin\psi - (\eta^2)^{-1/2}\eta_0\cos\psi = \psi.$$

Squaring and adding would give $\xi_0^2 + \eta_0^2/\eta^2 = 1 + \psi^2 \leqslant 1$, and thus $\psi = 0$. According to (2.4), we have $\xi_0 < 1$ on N_-. Moreover, we have

$$q^2 = \big(\eta^2/(m\alpha)\big)^2\big(1 - \xi_0\cos\psi - (\eta^2)^{-1/2}\eta_0\sin\psi\big)^2 > 0$$

and

$$H = p^2/(2m) - \alpha/q = -m\alpha^2/(2\eta^2) < 0.$$

Thus, if we identify Σ_- with its image under the coordinate mapping in $\boldsymbol{R}^3 \times \boldsymbol{R}^3$, (2.7) describes a C^∞ mapping $G\colon N_- \to \Sigma_-$. The proof of the relations $G \circ F = \mathrm{id}_{\Sigma_-}$ and $F \circ G = \mathrm{id}_{N_-}$ follows by an elementary calculation. ∎

Thus, Σ_- is embedded diffeomorphically in M_- by means of (2.1). The complement of the image N_- of Σ_- in M_- is N_∞. Now, according to (2.3), the Hamilton function H can be extended to a C^∞ function defined on all of M_-. On the other hand, according to (2.7), $\vec{q}$ would go to zero on N_∞, while $\vec{p}$ would diverge there in such a way that $H = p^2/2m - \alpha/q$ would remain finite. Thus N_∞ corresponds to a family of two-dimensional spheres of infinite momenta, parametrized by the positive quantity $(\eta^2)^{1/2}$ or, equivalently, by the values of the negative energy $-H$. In the following, we will show that the addition of N_∞ to the actual phase space N_- regularizes the Kepler problem in the sense of Moser [9], that is, that the vector field of H, extended to all of M_-, is complete and thus generates a global one-parameter group of transformations on M_-, which describes the dynamics of the Kepler problem. In addition, we will show that the SO(4) Lie algebra of Hamiltonian vector fields, which is generated by the functions L_i^-, K_i^-, $i = 1, 2, 3$, on Σ_-, can be extended to all of M_-, is complete there, and thus generates a global Hamiltonian action of the symmetry group SO(4) on the regularized phase space.

First, we will study the transferral of the symplectic structure under the mapping F: $\Sigma_- \to M_-$. We will use $\bar{\xi}_\alpha, \bar{\eta}_\alpha$, $\alpha = 0, 1, 2, 3$, to denote the coordinate functions on $\boldsymbol{R}^4 \times \boldsymbol{R}^4$ which are defined by

$$\bar{\xi}_\alpha(\xi, \eta) = \xi_\alpha, \qquad \bar{\eta}_\alpha(\xi, \eta) = \eta_\alpha.$$

The restriction of these functions to M_- will be denoted by

$$\hat{\xi}_\alpha := \bar{\xi}_\alpha | M_-, \qquad \hat{\eta}_\alpha := \bar{\eta}_\alpha | M_-.$$

If j: $M_- \to \boldsymbol{R}^4 \times \boldsymbol{R}^4$ denotes the identical embedding, then we have $\hat{\xi}_\alpha = j^* \bar{\xi}_\alpha = \bar{\xi}_\alpha \circ j$, $\hat{\eta}_\alpha = j^* \bar{\eta}_\alpha = \bar{\eta}_\alpha \circ j$. On $\boldsymbol{R}^4 \times \boldsymbol{R}^4$, we define a symplectic structure by means of the closed 2-form

$$\bar{\omega} := \sum_{\alpha=0}^{3} d\bar{\xi}_\alpha \wedge d\bar{\eta}_\alpha.$$

The pullback $\hat{\omega} := j^* \bar{\omega}$ of this 2-form to M_- under j is a closed, non-degenerate 2-form on M_-. Thus $(M_-, \hat{\omega})$ is a symplectic manifold. Since j^* is a homomorphism of the algebra of alternating forms which commutes with the exterior derivation, we have

$$\hat{\omega} = j^* \bar{\omega} = \sum_{\alpha=0}^{3} d\hat{\xi}_\alpha \wedge d\hat{\eta}_\alpha. \tag{2.8}$$

If, instead of the restriction $\omega | \Sigma_-$ of the canonical symplectic form ω on the phase space T^*Q, we simply write ω, according to the agreement at the beginning of this chapter for the coordinates q_i, p_i, then (Σ_-, ω) is a symplectic manifold.

THEOREM 2. *$\omega = F^*(\hat{\omega} | N_-)$, that is, F is a symplectomorphism from (Σ_-, ω) onto $(N_-, \hat{\omega} | N_-)$.*

Proof: Since N_- is an open submanifold of M_-, $(N_-, \hat{\omega}|N_-)$ is a symplectic manifold. According to Theorem 1, the functions ξ_α, η_α on Σ_- and $\hat{\xi}_\alpha|N_-$, $\hat{\eta}_\alpha|N_-$ on N_-, $\alpha = 0, 1, 2, 3$, are related by the equations $\xi_\alpha = F^*(\hat{\xi}_\alpha|N_-)$, $\eta_\alpha = F^*(\hat{\eta}_\alpha|N_-)$. Thus, for the 1-forms defined on Σ_- and M_- by

$$\nu := \sum_{\alpha=0}^{3} \xi_\alpha d\eta_\alpha = \xi \cdot d\eta, \qquad \nu := \sum_{\alpha=0}^{3} \hat{\xi}_\alpha d\hat{\eta}_\alpha = \hat{\xi} \cdot d\hat{\eta}, \tag{2.9}$$

we have

$$\begin{aligned} F^*(\hat{\nu}|N_-) &= F^*(\hat{\xi}|N_- \cdot d\hat{\eta}|N_-) = F^*(\hat{\xi}|N_-) \cdot F^*(d\hat{\eta}|N_-) \\ &= \xi \cdot d\big(F^*(\hat{\eta}|N_-)\big) = \xi \cdot d\eta = \nu. \end{aligned}$$

In order to calculate ν, we write (2.1) in the form

$$\begin{aligned} \xi &= a\sin\varphi + b\cos\varphi, \qquad \eta = v(-a\cos\varphi + b\sin\varphi), \\ a &:= \begin{bmatrix} \vec{q}\cdot\vec{p}/v \\ \vec{q}/q - \vec{q}\cdot\vec{p}\vec{p}/(m\alpha) \end{bmatrix}, \qquad v := m\alpha/(-2mH)^{1/2}, \\ b &:= \begin{bmatrix} qp^2/(m\alpha) - 1 \\ q\vec{p}/v \end{bmatrix}, \qquad \varphi := \vec{q}\cdot\vec{p}/v. \end{aligned} \tag{2.10}$$

Because of $a^2 = 1 = b^2$, $a \cdot b = 0$, we have $a \cdot da = 0 = b \cdot db$, $a \cdot db + b \cdot da = 0$, and thus $\nu = \xi \cdot d\eta = v(d\varphi + a \cdot db) = \vec{q} \cdot d\vec{p} + d(\vec{q} \cdot \vec{p})$. Thus we have

$$\omega = \sum_{i=1}^{3} dq_i \wedge dp_i = d\nu = d\big(F^*(\hat{\nu}|N_-)\big) = F^* d(\hat{\nu}|N_-) = F^*(d\hat{\nu}|N_-) = F^*(\hat{\omega}|N_-). \quad \blacksquare$$

On the ground of this theorem, we can consider F as a symplectic embedding of (Σ_-, ω) into the symplectic manifold $(M_-, \hat{\omega})$.

We now describe global actions of the groups SO(4) and $\boldsymbol{R} := (\boldsymbol{R}, +)$ on the manifold M_-. We will see that the action of $\boldsymbol{R}$ can be interpreted as the temporal evolution and the action of SO(4) as the global symmetry of the Kepler problem. To see this, we will calculate the vector fields on M_- belonging to the infinitesimal transformations of these actions and show that they are Hamiltonian, and that the restrictions of the corresponding Hamilton functions to the submanifold N_- are mapped, by means of the pullback F^*, onto the constants of motion L_i^-, K_i^- and the Hamiltonian function of the Kepler problem which were given in the first chapter.

We define an action σ: $\mathrm{SO}(4) \times M_- \to M_-$ of the group SO(4) on M_- by means of $\sigma\big(R, (\xi, \eta)\big) := (R\xi, R\eta)$, whereby $\xi \mapsto R\xi$ denotes the canonical action of SO(4) on $\boldsymbol{R}^4$. We let $\boldsymbol{R}$ operate on M_- by τ: $\boldsymbol{R} \times M_- \to M_-$, $\tau\big(t, (\xi, \eta)\big) := \big(\xi\cos\omega_\eta t + \eta(\eta^2)^{-1/2}\sin\omega_\eta t, \eta\cos\omega_\eta t - \xi(\eta^2)^{1/2}\sin\omega_\eta t\big)$, $\omega_\eta := m\alpha^2(\eta^2)^{-3/2}$.

THEOREM 3. $\big(\mathrm{SO}(4), \sigma\big)$ *and* $(\boldsymbol{R}, \tau)$ *are Lie transformation groups of* M_-. *The orbits of* SO(4) *under* σ *are the 5-dimensional submanifolds* M_-^r *characterized by* $(\eta^2)^{1/2} = r$,

$r \in \boldsymbol{R}_{+}^{\times} = \{x \in \boldsymbol{R} |\ x > 0\}$. *The actions σ and τ commute. Any orbit in M_- of $\boldsymbol{R}$ under τ is completely contained in a submanifold M_-^r.*

Proof: The first assertion holds because: (1) the mappings σ: $\mathrm{SO}(4) \times M_- \to M_-$ and τ: $\boldsymbol{R} \times M_- \to M_-$, considered as mappings of manifolds, are of class C^∞, and (2) the transformation group axioms $\sigma(R', \sigma(R, \cdot)) = \sigma(R'R, \cdot)$ and $\tau(t', \tau(t, \cdot)) = \tau(t' + t, \cdot)$ hold. The first property may be proven using charts, which we shall not do here. The second property follows by an elementary calculation.

Since R is an orthogonal transformation, we have $(R\eta)^2 = \eta^2$. Thus every submanifold M_-^r is invariant under σ. In order to prove that SO(4) acts transitively on M_-^r under σ, we show that any point $(\xi, \eta) \in M_-^r$ can be transformed to the point $(\mathring{\xi}, \mathring{\eta}) = ((1, 0, 0, 0), (0, 0, 0, r))$ by an element of SO(4). Since SO(4) acts transitively on the 3-sphere S^3, there is an $R_1 \in \mathrm{SO}(4)$ such that $R_1 \xi = \mathring{\xi}$. The stability subgroup of SO(4) in the point $\mathring{\xi}$ is formed by the elements of the form

$$\mathring{R} = \begin{bmatrix} 1 & 0 \\ 0 & \hat{R} \end{bmatrix} \quad \text{with} \quad \hat{R} \in \mathrm{SO}(3).$$

Since $\eta \cdot \xi = R_1 \xi \cdot R_1 \eta = \mathring{\xi} \cdot R_1 \eta = 0$, $R_1 \eta$ has the form $(0, \vec{\eta}_1)$ with $\vec{\eta}_1^2 = r^2$, that is, $\vec{\eta}_1$ lies on the 2-sphere S_r^2 of radius r. Since SO(3) acts transitively on every such 2-sphere, we can choose an $\hat{R}_2 \in \mathrm{SO}(3)$ that transforms $\vec{\eta}_1$ into $\hat{R}_2 \vec{\eta}_1 = (0, 0, r) = \mathring{\vec{\eta}}$. But then we have $\sigma(\mathring{R}_2 R_1, (\xi, \eta)) = (\mathring{\xi}, \mathring{\eta})$.

The last two assertions follow from an elementary calculation. ∎

Clearly, one can identify the manifold M_-^r with the sphere-bundle $(T^*S^3)_r$ on the 3-sphere S^3, whose fibers are the 2-spheres S_r^2 of radius r in the cotangent planes. M_- can then be thought of as the union $\bigcup_{r>0} M_-^r$ of all orbits of SO(4) under σ.

The Lie algebra of SO(4) is formed by the antisymmetric, real 4×4 matrices. The following matrices form a basis of it:

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2.11}$$

$$B_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

They obey the following commutation relations:

$$[A_i, A_j] = \sum_{k=1}^{3} \varepsilon_{ijk} A_k, \qquad [A_i, B_j] = \sum_{k=1}^{3} \varepsilon_{ijk} B_k,$$
$$[B_i, B_j] = \sum_{k=1}^{3} \varepsilon_{ijk} A_k. \tag{2.12}$$

We consider the following one-parameter subgroups of SO(4):

$$\left\{ e^{s\vec{n}\cdot\vec{A}} = \begin{bmatrix} 0 & \vec{O} \\ \vec{O} & \mathbf{1}_3 \cos s + (1-\cos s)\vec{n}\vec{n} + \sin s\vec{n}\times \end{bmatrix} \middle| \; |\vec{n}| = 1 \;,\; s \in \boldsymbol{R} \right\},$$
$$\left\{ e^{s\vec{n}\cdot\vec{B}} = \begin{bmatrix} \cos s & \vec{n}\sin s \\ -\vec{n}\sin s & \mathbf{1}_3 - (1-\cos s)\vec{n}\vec{n} \end{bmatrix} \middle| \; |\vec{n}| = 1, \; s \in \boldsymbol{R} \right\}. \tag{2.13}$$

In order to determine the vector fields $\vec{n}\cdot\vec{A}^*$ and $\vec{n}\cdot\vec{B}^*$ that these subgroups induce on M_- by means of σ, we calculate the corresponding Lie derivations of $\mathscr{F}M_-$:

$$(\mathscr{L}_{\vec{n}\cdot\vec{A}^*} f)(\xi, \eta) := \lim_{s\to 0} (1/s)\left[f\circ\sigma\left(e^{s\vec{n}\cdot\vec{A}}, (\xi, \eta)\right) - f(\xi, \eta)\right], \tag{2.14}$$
$$(\mathscr{L}_{\vec{n}\cdot\vec{B}^*} f)(\xi, \eta) := \lim_{s\to 0} (1/s)\left[f\circ\sigma\left(e^{s\vec{n}\cdot\vec{B}}, (\xi, \eta)\right) - f(\xi, \eta)\right].$$

For this, we introduce an atlas $\{(U^\varepsilon, k^\varepsilon) |\; \varepsilon = \pm\}$ on M_-, consisting of the charts

$$U^\varepsilon = \{(\xi, \eta) \in M_- |\; 1 + \varepsilon\xi_0 > 0\}, \qquad \varepsilon = \pm, \tag{2.15}$$

and the coordinate mappings

$$k^\varepsilon\colon U^\varepsilon \to \boldsymbol{R}^6, \qquad x_i^\varepsilon(\xi, \eta) = \xi_i/(1+\varepsilon\xi_0),$$
$$y_i^\varepsilon(\xi, \eta) = \eta_i(1+\varepsilon\xi_0) - \xi_i\eta_0, \qquad i = 1, 2, 3, \; \varepsilon = \pm. \tag{2.16}$$

Here, the x_i^ε and y_i^ε are the composition of k^ε with the ith and $(i+3)$rd coordinate projections in $\boldsymbol{R}^6$, respectively. We will also consolidate the components x_i^ε and y_i^ε to two $\boldsymbol{R}^3$-valued functions $\vec{x}^\varepsilon$ and $\vec{y}^\varepsilon$, respectively, and use the Euclidean scalar product "$\cdot$" and norm squared. The image of U^ε in $\boldsymbol{R}^6$ is the set $V = \{(x_i, y_i) \in \boldsymbol{R}^6 |\; \vec{y}^2 \neq 0\}$. The inverse of (2.16) is then defined on V by

$$\xi_0 = \varepsilon \frac{1 - (\vec{x}^\varepsilon(\xi, \eta))^2}{1 + (\vec{x}^\varepsilon(\xi, \eta))^2}, \qquad \xi_i = \frac{2x_i^\varepsilon(\xi, \eta)}{1 + (\vec{x}^\varepsilon(\xi, \eta))^2},$$
$$\eta_i = \frac{1 + (\vec{x}^\varepsilon(\xi, \eta))^2}{2} y_i^\varepsilon(\xi, \eta) - \vec{x}^\varepsilon(\xi, \eta)\cdot\vec{y}^\varepsilon(\xi, \eta) x_i^\varepsilon(\xi, \eta), \tag{2.17}$$
$$\eta_0 = -\varepsilon\vec{x}^\varepsilon(\xi, \eta)\cdot\vec{y}^\varepsilon(\xi, \eta), \qquad i = 1, 2, 3.$$

In $U_-\cap U_+$ we have

$$x_i^{-\varepsilon} = x_i^\varepsilon/(\vec{x}^\varepsilon)^2, \qquad y_i^{-\varepsilon} = (\vec{x}^\varepsilon)^2 y_i^\varepsilon - 2\vec{x}^\varepsilon\cdot\vec{y}^\varepsilon x_i^\varepsilon, \qquad i = 1, 2, 3. \tag{2.18}$$

For the 1-form $\hat{\nu}$ from (2.9) and the symplectic form $\hat{\omega}$ from (2.8) we then have

$$\hat{\nu}|U^\varepsilon = \vec{x}^\varepsilon \cdot d\vec{y}^\varepsilon, \quad \hat{\omega}|U^\varepsilon = \sum_{i=1}^{3} dx_i^\varepsilon \wedge dy_i^\varepsilon, \tag{2.19}$$

that is, the atlas chosen consists of symplectic charts. The Poisson bracket of two functions $f, g \in \mathscr{F}M_-$ thus has, locally, the form

$$\{f, g\}|U^\varepsilon = \sum_{i=1}^{3} \left(\frac{\partial f}{\partial x_i^\varepsilon} \frac{\partial g}{\partial y_i^\varepsilon} - \frac{\partial g}{\partial x_i^\varepsilon} \frac{\partial f}{\partial y_i^\varepsilon} \right) \Big| U^\varepsilon, \tag{2.20}$$

whereby the partial derivatives are defined by

$$\frac{\partial f}{\partial x_i^\varepsilon}(\xi, \eta) := \partial_i \left(f \circ (k^\varepsilon)^{-1} \right) \left(k^\varepsilon(\xi, \eta) \right),$$

$$\frac{\partial f}{\partial y_i^\varepsilon}(\xi, \eta) := \partial_{i+3} \left(f \circ (k^\varepsilon)^{-1} \right) \left(k^\varepsilon(\xi, \eta) \right), \quad i = 1, 2, 3,$$

with the partial derivatives $\partial_1, \ldots, \partial_6$ in the coordinate directions of $\boldsymbol{R}^6$. Our choice of a symplectic atlas was inspired by the paper of J. Moser [9].

Thus, on U^ε, we obtain the following formulas for the Lie derivations from (2.14):

$$\begin{aligned} \mathscr{L}_{A_i^*} &= \sum_{j,k=1}^{3} \varepsilon_{ijk} \left(x_j^\varepsilon \frac{\partial}{\partial x_k^\varepsilon} + y_j^\varepsilon \frac{\partial}{\partial y_k^\varepsilon} \right), \\ \mathscr{L}_{B_i^*} &= -\varepsilon \left[\frac{1 - (\vec{x}^\varepsilon)^2}{2} \frac{\partial}{\partial x_i^\varepsilon} + x_i^\varepsilon \vec{x}^\varepsilon \cdot \frac{\vec{\partial}}{\partial x^\varepsilon} - \vec{x}^\varepsilon \cdot \vec{y}^\varepsilon \frac{\partial}{\partial y_i^\varepsilon} + \right. \\ &\qquad \left. + (y_i^\varepsilon \vec{x}^\varepsilon - x_i^\varepsilon \vec{y}^\varepsilon) \cdot \frac{\vec{\partial}}{\partial y^\varepsilon} \right], \quad i = 1, 2, 3. \end{aligned} \tag{2.21}$$

We define the functions L_i^*, K_i^* in $\mathscr{F}M_-$ by

$$L_i^*(\xi, \eta) := \sum_{j,k=1}^{3} \varepsilon_{ijk} \xi_j \eta_k, \quad K_i^*(\xi, \eta) := \eta_0 \xi_i - \xi_0 \eta_i. \tag{2.22}$$

For their partial derivatives, we obtain

$$\begin{aligned} \frac{\partial L_i^*}{\partial x_j^\varepsilon} &= \sum_{k=1}^{3} \varepsilon_{ijk} y_k^\varepsilon, \quad \frac{\partial L_i^*}{\partial y_j^\varepsilon} = -\sum_{k=1}^{3} \varepsilon_{ijk} x_k^\varepsilon, \\ \frac{\partial K_i^*}{\partial x_j^\varepsilon} &= \varepsilon (x_j^\varepsilon y_i^\varepsilon - x_i^\varepsilon y_j^\varepsilon - \vec{x}^\varepsilon \cdot \vec{y}^\varepsilon \delta_{ij}), \\ \frac{\partial K_i^*}{\partial y_j^\varepsilon} &= \varepsilon \left(\frac{(\vec{x}^\varepsilon)^2 - 1}{2} \delta_{ij} - x_i^\varepsilon x_j^\varepsilon \right). \end{aligned} \tag{2.23}$$

These serve to show, for one thing, that these functions form the Lie algebra of the group SO(4) under the Poisson bracket, that is, that the following relations hold:

$$\{L_i^*, L_j^*\} = \sum_{k=1}^{3} \varepsilon_{ijk} L_k^*, \qquad \{L_i^*, K_j^*\} = \sum_{k=1}^{3} \varepsilon_{ijk} K_k^*,$$
$$\{K_i^*, K_j^*\} = \sum_{k=1}^{3} \varepsilon_{ijk} L_k^*. \tag{2.24}$$

On the other hand, we can see that the Lie derivations in (2.21) can be written in the following way:

$$\mathscr{L}_{A_i^*} = \sum_{j=1}^{3} \left(\frac{\partial L_i^*}{\partial y_j^\varepsilon} \frac{\partial}{\partial x_j^\varepsilon} - \frac{\partial L_i^*}{\partial x_j^\varepsilon} \frac{\partial}{\partial y_j^\varepsilon} \right),$$
$$\mathscr{L}_{B_i^*} = \sum_{j=1}^{3} \left(\frac{\partial K_i^*}{\partial y_j^\varepsilon} \frac{\partial}{\partial x_j^\varepsilon} - \frac{\partial K_i^*}{\partial x_j^\varepsilon} \frac{\partial}{\partial y_j^\varepsilon} \right), \qquad i = 1, 2, 3. \tag{2.25}$$

The last equations mean that the vector fields A_i^* and B_i^* are Hamiltonian and are derived from the Hamilton functions L_i^* and K_i^*.

We can now carry out the same analysis for the action τ of $\boldsymbol{R}$ on M_-. Let C denote the basis element of the Lie algebra of $\boldsymbol{R}$ for which $\exp(tC) = t \in \boldsymbol{R}$ holds. The induced vector field C^* on M_- is then given by the Lie derivation

$$(\mathscr{L}_{C^*} f)(\xi, \eta) = \lim_{t \to 0} (1/t) \left[f \circ \tau\big(t, (\xi, \eta)\big) - f(\xi, \eta) \right]$$
$$= \frac{4m\alpha^2}{(\vec{y}^\varepsilon)^2 (1 + (\vec{x}^\varepsilon)^2)^2} \left[\frac{\vec{y}^\varepsilon}{(\vec{y}^\varepsilon)^2} \cdot \frac{\partial}{\partial \vec{x}^\varepsilon} - \frac{2\vec{x}^\varepsilon}{1 + (\vec{x}^\varepsilon)^2} \cdot \frac{\partial}{\partial \vec{y}^\varepsilon} \right] f(\xi, \eta). \tag{2.26}$$

We define a function H^* in M_- by

$$H^*(\xi, \eta) = -m\alpha^2 / (2\eta^2). \tag{2.27}$$

For its partial derivatives, we obtain

$$\frac{\partial H^*}{\partial x_i^\varepsilon} = \frac{8m\alpha^2 x_i^\varepsilon}{(\vec{y}^\varepsilon)^2 (1 + (\vec{x}^\varepsilon)^2)^3}, \qquad \frac{\partial H^*}{\partial y_i^\varepsilon} = \frac{4m\alpha^2 y_i^\varepsilon}{((\vec{y}^\varepsilon)^2)^2 (1 + (\vec{x}^\varepsilon)^2)^2}. \tag{2.28}$$

Thus, the Lie derivation can be written in the form

$$\mathscr{L}_{C^*} = \sum_{j=1}^{3} \left(\frac{\partial H^*}{\partial y_j^\varepsilon} \frac{\partial}{\partial x_j^\varepsilon} - \frac{\partial H^*}{\partial x_j^\varepsilon} \frac{\partial}{\partial y_j^\varepsilon} \right), \tag{2.29}$$

from which it follows that H^* is a Hamilton function for the vector field C^*. Finally, we can verify that H^* commutes with the functions L_i^*, K_i^* under the Poisson bracket:

$$\{L_i^*, H^*\} = 0, \qquad \{K_i^*, H^*\} = 0, \qquad i = 1, 2, 3. \tag{2.30}$$

We summarize the last results in

THEOREM 4. *The actions σ of* SO(4) *and τ of $\boldsymbol{R}$ on M_- defined before Theorem* 3 *are globally Hamiltonian, that is, the vector fields on M_- generated by the one parameter subgroups are globally Hamiltonian. The basis elements A_i, B_i, $i = 1, 2, 3$* (cf. (2.11)), *of the Lie algebra of* SO(4) *and C of the Lie algebra of $\boldsymbol{R}$ correspond to the Hamilton functions L_i^*, K_i^*, $i = 1, 2, 3$, from* (2.22) *and H^* from* (2.27), *respectively.*

The connection between Theorem 4 and the Kepler problem is given by the following theorem.

THEOREM 5. *The functions L_i^*, K_i^*, $i = 1, 2, 3$, and H^* on M_- are extensions of the components L_i^- of the angular momentum, K_i^- of the modified Runge–Lenz vector and of the energy H of the Kepler problem* (cf. (1.1), (1.2), (1.11)), *respectively. More exactly, we have:*

$$L_i^*|N_- \circ F = L_i^-, \qquad K_i^*|N_- \circ F = K_i^-, \qquad H^*|N_- \circ F = H,$$

whereby $F\colon \Sigma_- \to M_-$ is the symplectic embedding described in Theorems 1 *and* 2.

Proof: The proof consists of a simple calculation. For example, (2.1) implies

$$H^*\big(\xi(\vec{q},\vec{p}), \eta(\vec{q},\vec{p})\big) = -m\alpha^2/\big(2\big(\eta(\vec{q},\vec{p})\big)^2\big) = p^2/(2m) - \alpha/q = H(q,p). \;\blacksquare$$

On the basis of these five theorems, we can think of M_- as a regularized model of the negative energy part Σ_- of the phase space of the Kepler problem. Both the temporal evolution, which is given by the action τ of $\boldsymbol{R}$ and the symmetry, which is given by the action σ of SO(4) are described globally. According to Theorem 5, the orbits M_-^r of SO(4) under σ can be thought of as regularizations of the energy hypersurfaces Σ_E with energy $E = -m\alpha^2/2r^2$ in phase space. Since SO(4) acts globally on M_-^r and the singular submanifold N_∞ intersects every M_-^r according to (2.4), the action of SO(4) on Σ_- is certainly not globally defined. The temporal evolution is periodic. The orbits of the temporal evolution intersect N_∞ if and only if the initial conditions are such that either $\vec{\xi}$ and $\vec{\eta}$ are parallel or $\vec{\xi}$ or $\vec{\eta} = \vec{O}$. Equivalent to that is vanishing angular momentum $\vec{L} = \vec{O}$. Thus, the temporal evolution does not yield a global flow on Σ_-. The physical meaning of the orbits with $\vec{L} = \vec{O}$ is that of collision orbits, which go into the origin of configuration space. The regularized model gives the picture of a periodic reflection at the origin as the spatial motion in a collision orbit.

We add a note on the transcendental equation (2.5). If we let the operation τ of $\boldsymbol{R}$ act on a point $(\xi, \eta) \in M_-$, then the angle evolves, using (2.5), according to the equation

$$\begin{aligned}\psi(t) &= \xi_0(t)\sin\psi(t) - (\eta^2)^{-1/2}\eta_0(t)\cos\psi(t)\\ &= \xi_0\sin\big(\psi(t)+\omega_\eta t\big) - (\eta^2)^{-1/2}\eta_0\cos\big(\psi(t)+\omega_\eta t\big),\end{aligned}$$

or, if we put

$$\xi_0 = (\xi_0^2+\eta_0^2/\eta^2)^{1/2}\cos\omega_\eta t_0, \qquad \eta_0 = (\eta^2\xi_0^2+\eta_0^2)^{1/2}\sin\omega_\eta t_0$$

with a suitable constant t_0:

$$\psi(t) = (\xi_0^2+\eta_0^2/\eta^2)^{1/2}\sin\big(\psi(t)+\omega_\eta(t-t_0)\big).$$

If we set $\psi(t)+\omega_\eta(t-t_0) =: u(t)$ and notice that $0 \leqslant \xi_0^2+\eta_0^2/\eta^2 = 1+2HL^2/m\alpha^2 \leqslant 1$ and $\omega_\eta = (-2mH)^{3/2}/m^2\alpha$, then we obtain the Kepler equation for the excentric anomaly u:

$$(-2mH)^{3/2}/(m^2\alpha)\,(t-t_0) = u(t)-(1+2HL^2/m\alpha^2)^{1/2}\sin u(t), \qquad u(t_0) = 0. \tag{2.31}$$

Thus ψ describes the aberration of the excentric from the mean anomaly. Equation (2.5) is thus to be interpreted as a generalization of the Kepler equation.

3. The positive energy case

In this section we treat the positive energy part Σ_+ of the phase space of the Kepler problem. Since the proofs and calculations are largely parallel to those of the previous section, we will restrict ourselves to the discussion of technical peculiarities of the theorems. As in the last section, we will think of the functions q_i, p_i and H as well as the symplectic form $\omega = \sum_{i=1}^{3} dq_i \wedge dp_i$ as being defined only on Σ_+ in order to avoid the notation of restrictions. We will often use the Minkowski scalar product in $\boldsymbol{R}^4$ in this section. In order to avoid confusion with the Euclidean scalar product, which we denote by a dot, we will use an asterisk for the Minkowski scalar product and define it by

$$x*y := x_0y_0-\sum_{k=1}^{3} x_ky_k = x_0y_0-\vec{x}\cdot\vec{y}, \qquad x_*^2 := x*x, \ x, y \in \boldsymbol{R}^4. \tag{3.1}$$

We consider the following functions on Σ_+:

$$\begin{aligned}
\xi_0 &= -\frac{(2mH)^{1/2}}{m\alpha}\vec{q}\cdot\vec{p}\sinh\chi+\left(\frac{qp^2}{m\alpha}-1\right)\cosh\chi, \\
\vec{\xi} &= \left(\frac{\vec{q}}{q}-\frac{\vec{q}\cdot\vec{p}}{m\alpha}\vec{p}\right)\sinh\chi+\frac{(2mH)^{1/2}}{m\alpha}\,q\vec{p}\cosh\chi, \\
\eta_0 &= \vec{q}\cdot\vec{p}\cosh\chi-\frac{m\alpha}{(2mH)^{1/2}}\left(\frac{qp^2}{m\alpha}-1\right)\sinh\chi, \\
\vec{\eta} &= -\frac{m\alpha}{(2mH)^{1/2}}\left(\frac{\vec{q}}{q}-\frac{\vec{q}\cdot\vec{p}}{m\alpha}\vec{p}\right)\cosh\chi-q\vec{p}\sinh\chi, \\
\chi &:= (2mH)^{1/2}/(m\alpha)\vec{q}\cdot\vec{p}.
\end{aligned} \tag{3.2}$$

We define a six-dimensional C^∞-submanifold M_+ of $\boldsymbol{R}^4\times\boldsymbol{R}^4$ by

$$M_+ := \{(\xi,\eta)\in\boldsymbol{R}^4\times\boldsymbol{R}^4 \mid \xi_*^2 = 1,\ \xi_0 > 0,\ \xi*\eta = 0,\ \eta_*^2 < 0\}. \tag{3.3}$$

The equations in (3.3) tell us that ξ lies in the forward half H^3 of the two-sheeted unit hyperboloid $\{\xi_*^2 = 1\}$ in $\boldsymbol{R}^4$, that η is tangential to H^3 in the point ξ and that η is not zero. M_+ is thus diffeomorphic to the total space $(T^*H^3)^\times$ of the cotangent bundle on H^3 with the zero section removed. Since H^3 is diffeomorphic to $\boldsymbol{R}^3$, M_+ is also diffeomorphic to $\boldsymbol{R}^3\times(\boldsymbol{R}^3\setminus\{0\})$. We define two C^∞-submanifolds of M_+ by

$$N'_\infty := \{(\xi, \eta) \in M_+ |\ \xi_0 = 1\}, \quad N_+ := M_+ \setminus N'_\infty. \tag{3.4}$$

N'_∞ is thus diffeomorphic to the punctured cotangent space on H^3 at the point $\xi = (1, \vec{O})$ and thus also to $\boldsymbol{R}^3 \setminus \{0\}$. N_+ is open and dense in M_+.

THEOREM 6. *The functions defined in* (3.2) *map* Σ_+ *diffeomorphically onto* N_+.

The condition $\xi_0 > 0$ requires some comment, and can be proven as follows:

$$\begin{aligned}
\xi_0 &= \left(\frac{qp^2}{m\alpha} - 1\right)\cosh\chi - \frac{(2mH)^{1/2}}{m\alpha}\,\vec{q}\cdot\vec{p}\sinh\chi \\
&\geqslant \left(1 + \frac{2qH}{\alpha}\right)\cosh\chi - \frac{(2mH)^{1/2}}{m\alpha}\,qp\cosh\chi \\
&= [(1+2Hq^2p^2/m\alpha^2)^{1/2} - (2Hq^2p^2/m\alpha^2)^{1/2}]\cosh\chi > 0.
\end{aligned}$$

The proof of the theorem uses the inverse mapping from N_+ onto Σ_+, which is given by

$$\begin{aligned}
\vec{q} &= \frac{-\eta_*^2}{m\alpha}[-(\xi_0 - \cosh\varrho)(-\eta_*^2)^{-1/2}\vec{\eta} + ((-\eta_*^2)^{-1/2}\eta_0 + \sinh\varrho)\vec{\xi}], \\
\vec{p} &= \frac{m\alpha\,[(-\eta_*^2)^{-1/2}\vec{\eta}\sinh\varrho + \vec{\xi}\cosh\varrho]}{(-\eta_*^2)^{1/2}[\xi_0\cosh\varrho + (-\eta_*^2)^{-1/2}\eta_0\sinh\varrho - 1]}, \\
\varrho &= \xi_0\sinh\varrho + (-\eta_*^2)^{-1/2}\eta_0\cosh\varrho.
\end{aligned} \tag{3.5}$$

The last equation, which defines ϱ implicitly as a function of (ξ, η), is the analog of the generalized Kepler equation in (2.5). The proof of the existence of a solution of this equation uses the inequality

$$\xi_0^2 - \eta_0^2/(-\eta_*^2) \geqslant \xi_0^2 - \eta_0^2/(-\eta_*^2) - (\vec{\xi}\times\vec{\eta})^2/(-\eta_*^2) = 1.$$

We define the functions $\hat{\xi}_\alpha$, $\hat{\eta}_\alpha$, $\alpha = 0, 1, 2, 3$, in $\mathscr{F}M_+$ as the restriction to M_+ of the coordinate functions on $\boldsymbol{R}^4 \times \boldsymbol{R}^4$. Then we can define a closed, non-degenerate 2-form $\hat{\omega}$ on M_+ by

$$\hat{\omega} := -d\hat{\xi}_0 \wedge d\hat{\eta}_0 + \sum_{i=1}^{3} d\hat{\xi}_i \wedge d\hat{\eta}_i. \tag{3.6}$$

We define 1-forms ν on Σ_+ and $\hat{\nu}$ on M_+ by

$$\nu := -\xi * d\eta, \quad \hat{\nu} := -\hat{\xi} * d\hat{\eta}. \tag{3.7}$$

Then we have $\hat{\omega} = d\hat{\nu}$. $(M_+, \hat{\omega})$ is a symplectic manifold. By means of a calculation similar to the one in (2.10)ff., one can show that $\nu = \vec{q}\cdot d\vec{p} + d(\vec{q}\cdot\vec{p})$ holds and thus $\omega = d\nu = F^*(d\hat{\nu}|N_+) = F^*(\hat{\omega}|N_+)$, whereby $F\colon \Sigma_+ \to M_+$ is the embedding from Theorem 6.

THEOREM 7. $\omega = F^*(\hat{\omega}|N_+)$, *i.e.* F *is a symplectomorphism from* (Σ_+, ω) *onto* $(N_+, \hat{\omega}|N_+)$.

We now define global actions σ and τ of the groups $\mathrm{SO}(1,3)_0$ and $\boldsymbol{R} := (\boldsymbol{R}, +)$ on M_+ by

$$\sigma(R, (\xi, \eta)) := (R\xi, R\eta), \qquad R \in \mathrm{SO}(1,3)_0$$

and

$$\tau(t, (\xi, \eta)) := (\xi \cosh\omega_\eta t - \eta(-\eta_*^2)^{-1/2} \sinh\omega_\eta t, \eta\cosh\omega_\eta t - \xi(-\eta_*^2)^{1/2} \sinh\omega_\eta t),$$
$$t \in \boldsymbol{R}, \qquad \omega_\eta := m\alpha^2(-\eta_*^2)^{-3/2},$$

respectively. Then the next theorem is parallel to Theorem 3.

THEOREM 8. $(\mathrm{SO}(1,3)_0, \sigma)$ *and* $(\boldsymbol{R}, \tau)$ *are* C^∞ *Lie transformation groups of* M_+. *The orbits of* $\mathrm{SO}(1,3)_0$ *under* σ *in* M_+ *are the 5-dimensional* C^∞ *submanifolds* M_+^r, *which are parametrized by* $(-\eta_*^2)^{1/2} = r$, $r \in \boldsymbol{R}_+^\times = \{x \in \boldsymbol{R} \mid x > 0\}$. *The actions* σ *and* τ *commute. Any orbit of* $\boldsymbol{R}$ *under* τ *in* M_+ *is completely contained in a submanifold* M_+^r.

We can think of the hypersurface M_+^r in phase space as a 2-sphere bundle over H^3. It is diffeomorphic to $\boldsymbol{R}^3 \times S^2$.

As a basis of the Lie algebra $\mathrm{SO}(1,3)_0$, we choose the following matrices:

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.8}$$

$$B_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad B_3 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

They obey the commutation relations

$$[A_i, A_j] = \sum_{k=1}^{3} \varepsilon_{ijk} A_k, \qquad [A_i, B_j] = \sum_{k=1}^{3} \varepsilon_{ijk} B_k, \qquad [B_i, B_j] = -\sum_{k=1}^{3} \varepsilon_{ijk} A_k. \tag{3.9}$$

The one parameter subgroups defined by

$$\begin{aligned} &\left\{ e^{s\vec{n}\cdot\vec{A}} = \begin{bmatrix} O & \vec{O} \\ \vec{O} & \mathbf{1}_3 \cos s + (1-\cos s)\vec{n}\vec{n} + \sin s \vec{n} x \end{bmatrix} \middle| \; |\vec{n}| = 1, \; s \in \boldsymbol{R} \right\}, \\ &\left\{ e^{s\vec{n}\cdot\vec{B}} = \begin{bmatrix} \cosh s & -\vec{n}\sinh s \\ -\vec{n}\sinh s & \mathbf{1}_3 + (\cosh s - 1)\vec{n}\vec{n} \end{bmatrix} \middle| \; |\vec{n}| = 1, \; s \in \boldsymbol{R} \right\}, \end{aligned} \tag{3.10}$$

generate the vector fields $\vec{n}\cdot\vec{A}^*$, $\vec{n}\cdot\vec{B}^*$ on M_+, which are induced by the action σ, analogous to (2.14). Since H^3 is contractible, we can define a symplectic coordinate system globally on M_+ by means of

$$(\vec{x}, \vec{y})\colon\ M_+ \to \boldsymbol{R}^3\times\boldsymbol{R}^3, \qquad \vec{x}(\xi, \eta) := \vec{\xi}, \qquad \vec{y}(\xi, \eta) := \vec{\eta}-\vec{\xi}\eta_0/\xi_0. \tag{3.11}$$

The inverse is

$$\begin{aligned} \xi_0 &= \left(1+\vec{x}(\xi, \eta)^2\right)^{1/2}, \qquad \vec{\xi} = \vec{x}(\xi, \eta), \\ \eta_0 &= \vec{x}(\xi, \eta)\cdot\vec{y}(\xi, \eta)\left(1+\vec{x}(\xi, \eta)^2\right)^{1/2}, \\ \vec{\eta} &= \vec{y}(\xi, \eta)+\vec{x}(\xi, \eta)\cdot\vec{y}(\xi, \eta)\vec{x}(\xi, \eta). \end{aligned} \tag{3.12}$$

One easily verifies that $\hat{\nu} = -\hat{\xi}*d\hat{\eta} = -\vec{y}\cdot d\vec{x}$, and $\hat{\omega} = d\hat{\nu} = d\vec{x}\wedge d\vec{y}$ hold and thus that (3.11) defines a global symplectic chart on $(M_+, \hat{\omega})$. Then we can calculate the Lie derivations corresponding to A_i^*, B_i^*, $i = 1, 2, 3$,

$$\begin{aligned} \mathscr{L}_{A_i^*} &= \sum_{j,k=1}^{3} \varepsilon_{ijk}\left(x_j\frac{\partial}{\partial x_k}+y_j\frac{\partial}{\partial y_k}\right), \\ \mathscr{L}_{B_i^*} &= -(1+\vec{x}^2)^{1/2}\frac{\partial}{\partial x_i}+y_i\frac{\vec{x}}{(1+\vec{x}^2)^{1/2}}\cdot\frac{\partial}{\partial \vec{y}}, \qquad i = 1, 2, 3. \end{aligned} \tag{3.13}$$

We define the functions L_i^*, K_i^*, $i = 1, 2, 3$, in $\mathscr{F}M_+$ by

$$\begin{aligned} L_i^*(\xi, \eta) &= \sum_{j,k=1}^{3} \varepsilon_{ijk}\xi_j\eta_k = \sum_{j,k=1}^{3} \varepsilon_{ijk}x_j(\xi, \eta)y_k(\xi, \eta), \\ K_i^*(\xi, \eta) &= \eta_0\xi_i-\xi_0\eta_i = -\left(1+\vec{x}(\xi, \eta)^2\right)^{1/2}y_i(\xi, \eta). \end{aligned} \tag{3.14}$$

Then we immediately obtain

$$\begin{aligned} \mathscr{L}_{A_i^*} &= \sum_{j=1}^{3}\left(\frac{\partial L_i^*}{\partial y_j}\frac{\partial}{\partial x_j}-\frac{\partial L_i^*}{\partial x_j}\frac{\partial}{\partial y_j}\right), \\ \mathscr{L}_{B_i^*} &= \sum_{j=1}^{3}\left(\frac{\partial K_i^*}{\partial y_j}\frac{\partial}{\partial x_j}-\frac{\partial K_i^*}{\partial x_j}\frac{\partial}{\partial y_j}\right). \end{aligned} \tag{3.15}$$

For the action τ of $\boldsymbol{R}$ on M_+, we have

$$\mathscr{L}_{C^*} = \sum_{j=1}^{3}\left(\frac{\partial H^*}{\partial y_j}\frac{\partial}{\partial x_j}-\frac{\partial H^*}{\partial x_j}\frac{\partial}{\partial y_j}\right), \tag{3.16}$$

whereby C is the element of the Lie algebra of $\boldsymbol{R}$ for which $\exp(tC) = t$ holds for all $t\in\boldsymbol{R}$ and H^* is the function in $\mathscr{F}M_+$ defined by

$$H^*(\xi, \eta) = \frac{m\alpha^2}{-2\eta_*^2} = \frac{m\alpha^2}{2[\vec{y}(\xi, \eta)^2+\left(\vec{x}(\xi, \eta)\cdot\vec{y}(\xi, \eta)\right)^2]}. \tag{3.17}$$

The Poisson brackets for the functions L_i^*, K_i^*, $i = 1, 2, 3$, and H^* are

$$\begin{aligned} \{L_i^*, L_j^*\} &= \sum_{k=1}^{3} \varepsilon_{ijk} L_k^*, \qquad \{L_i^*, K_j^*\} = \sum_{k=1}^{3} \varepsilon_{ijk} K_k^*, \\ \{K_i^*, K_j^*\} &= -\sum_{k=1}^{3} \varepsilon_{ijk} L_k^*, \qquad \{L_i^*, H^*\} = \{K_i^*, H^*\} = 0. \end{aligned} \tag{3.18}$$

The next theorem follows from the last results.

THEOREM 9. *The actions σ of* $SO(1, 3)_0$ *and τ of $\boldsymbol{R}$ on M_+ defined before Theorem* 8 *are globally Hamiltonian. The basis elements A_i, B_i, $i = 1, 2, 3$,* (cf. (3.9)) *and C of the Lie algebras of* $SO(1, 3)_0$ *and $\boldsymbol{R}$ correspond to the Hamilton functions L_i^*, K_i^*, $i = 1, 2, 3$, of* (3.14) *and H^* of* (3.17), *respectively.*

The following theorem is the result of a simple calculation.

THEOREM 10. *The functions L_i^*, K_i^*, $i = 1, 2, 3$, and H^* are extensions onto M_+ of the components L_i^+ of angular momentum, K^+ of the modified Runge–Lenz vector and the energy H of the Kepler problem, respectively, which are defined on Σ_+. That is, we have*

$$L_i^*|N_+ \circ F = L_i^+, \qquad K_i^*|N_+ \circ F = K_i^+, \qquad H^*|N_+ \circ F = H,$$

whereby $F\colon \Sigma_+ \to M_+$ is the symplectic embedding described in Theorems 6 *and* 7.

The commentary at the end of Section 2 also makes sense in the present case of positive energy. We want to emphasize that the symmetry group $SO(1, 3)_0$ operates transitively on the hypersurfaces M_+^r of constant energy and that every M_+^r meets the singular submanifold N'_∞. Therefore, for positive energies, the regularization also allows one to globalize the time flow and the symmetry operations simultaneously. In this case, Kepler's equation

$$\begin{aligned} (2mH)^{3/2}/(m^2\alpha)\,(t-t_0) &= (1+2HL^2/m\alpha^2)^{1/2}\sinh v(t) - v(t), \\ v(t_0) &= 0, \end{aligned} \tag{3.19}$$

follows from the last equation in (3.5) with the substitutions

$$\xi_0 = (\xi_0^2 - \eta_0^2/(-\eta_*^2))^{1/2}\cosh \omega_\eta t_0,$$

$$\eta_0 = (\xi_0^2(-\eta_*^2) - \eta_0^2)^{1/2}\sinh \omega_\eta t_0, \qquad \varrho(t) = v(t) + \omega_\eta(t-t_0).$$

4. Complementary remarks

In Sections 2 and 3, we have regularized the negative and positive energy parts of the phase space of the classical Kepler problem in such a way that both the time flow and the symmetry operations are realized in a global and simple way. In conclusion, we want to discuss two still open questions.

1. The phase space hypersurface Σ_0 of zero energy: It is diffeomorphic to $(\boldsymbol{R}^3 \setminus \{0\}) \times S^2$. Because of the possibility of collisions at $q = 0$, the Hamiltonian vector field does not generate a global flow here either. By adding a 2-sphere with infinite momentum to

Σ_0, one can achieve a regularization with a global time flow here as well. The regularized manifold M_0 is diffeomorphic to $\mathbf{R}^3 \times S^2$ and thus of the same diffeomorphism type as the regularized hypersurfaces of constant positive energy. One can easily make M_0 into a homogeneous space for the inhomogeneous rotation group ISO(3). We have also found a transitive action of ISO(3) on M_0 that commutes with the time flow, but we are not clear about the meaning of this within the context of Hamiltonian mechanics.

2. The action of the non-symmetry groups on the regularized phase space models M_- and M_+: As is known, every diffeomorphism of a manifold induces a symplectomorphism of the cotangent bundle of the manifold with its canonical symplectic structure (cf. Abraham–Marsden [1], p. 97).

Since such a symplectomorphism maps the cotangent spaces linearly, it leaves the zero section of the cotangent bundle invariant. Applied to our models, that means that the group of diffeomorphisms of S^3 (resp. H^3) induces canonical transformations of M_- (resp. M_+).

In the case of negative energy, we can let the de Sitter group $\mathrm{SO}(1,4)_0$ act as a group of diffeomorphisms of S^3 in such a way that the restriction to the subgroup SO(4) coincides with the symmetry operation described in Section 2. Every element of $\mathrm{SO}(1,4)_0$ can be written as a product of an SO(4) rotation and a Lorentz "boost" of the form

$$L_v = \begin{bmatrix} (1-v^2)^{-1/2} & v(1-v^2)^{-1/2} \\ v(1-v^2)^{-1/2} & \mathbf{1}_4 + \left((1-v^2)^{-1/2}-1\right)vv/v^2 \end{bmatrix}$$

with a four-component velocity $v \in \{v \in \mathbf{R}^4 |\ v^2 < 1\}$. The action of L_v on S^3 can then be defined by

$$\xi \mapsto \lambda_v(\xi) = \frac{1}{1+v\cdot\xi}\left[(1-v^2)^{1/2}\xi + \left(1+\frac{v\cdot\xi}{1+(1-v^2)^{1/2}}\right)v\right].$$

The form of this action is suggested by the well-known action of the Lorentz group $\mathrm{SO}(1,3)_0$ on the 2-sphere. The induced action of L_v on $(T^*S^3)^\times \simeq M_-$ is then described by the additional mapping

$$\eta \mapsto \lambda_v'(\xi)\eta = \frac{(1-v^2)^{1/2}}{1+v\cdot\xi}\left[\eta - \frac{\eta\cdot v}{1+v\cdot\xi}\left(\xi + \frac{v}{1+(1-v^2)^{1/2}}\right)\right].$$

Because of

$$(\lambda_v'(\xi)\eta)^2 = \frac{1-v^2}{(1+v\cdot\xi)^2}\,\eta^2 \quad \text{and} \quad 0 < \frac{1-v^2}{(1+v\cdot\xi)^2} < \infty,$$

this action is transitive on M_-. Since the action of L_v does not commute with the time flow, $\mathrm{SO}(1,4)_0$ operates as a non-symmetry group on M_-.

In the positive energy case, we have not been able to define an action of either of the de Sitter groups $\mathrm{SO}(1,4)_0$ or $\mathrm{SO}(2,3)_0$ on H^3 in such a way that the restriction to the subgroup $\mathrm{SO}(1,3)_0$ would be the symmetry operation described in Section 3.

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